# 13 The Measure of an Angle

As we have progressed through the first twelve sections, we have built up an axiom system piece by piece. Starting with the underlying set structure in an abstract geometry, we added the incidence axioms, the ruler postulate, and the plane separation axiom. So we can say that we had

Axiom 1: The Incidence Axiom

Axiom 2: The Ruler Postulate

Axiom 3: PSP

Now we want to add another axiom (Axiom 4: The Protractor Postulate) to make our geometry look more like the geometry that is studied in high school. In this section we shall define what is meant by an angle measure and indicate how angle measures are defined in our two basic models.

After we have an angle measure our geometry will look very much like "high school geometry." However, we will still be missing one important assumption. That assumption is the Side-Angle-Side (SAS) congruence axiom (Axiom 5: SAS). Without it, some results can occur which are very unusual for someone accustomed only to Euclidean geometry. In particular, we will see in examples that without SAS it is possible for the sum of the measures of the angles of a triangle to be greater than 180 degrees, or for the length of one side of a triangle to be longer than the sum of the other two.

**<u>Definition</u> (angle measure or protractor)** Let $r_0$ be a fixed positive real number. In a Pasch geometry, an angle measure (or protractor) based on $r_0$ is a function $m$ from the set $\mathcal{A}$ of all angles to the set of real numbers such that

(i) If $\measuredangle ABC \in \mathcal{A}$ then $0 < m(\measuredangle ABC) < r_0$;

(ii) If $\overrightarrow{BC}$ lies in the edge of the half plane $H_1$ and if $\theta$ is a real number with $0 < \theta < r_0$, then there is a unique ray $\overrightarrow{BA}$ with $A \in H_1$ and $m(\measuredangle ABC) = \theta$;

(iii) If $D \in \text{int}(\measuredangle ABC)$ then $m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC)$.

We should note that the definition of an angle measure does not even make sense unless we have PSA (or equivalently, Pasch) since we must use the idea of the interior of an angle. If $r_0 = 180$, $m$ is called degree measure. If $r_0 = \pi$, then $m$ is called radian measure. If $r_0 = 200$, then $m$ is called grade measure. Traditionally, degree measure is used in geometry. Radian measure is used in calculus because it makes the differentiation formulas most natural.

**<u>Convention</u>**. Except in Section "Euclidean and Poincaré Angle Measure", we shall always use degree measure ($r_0 = 180$) without further assumption.

**<u>Definition</u> (protractor geometry)** A protractor geometry $\{\mathcal{S}, \mathcal{L}, d, m\}$ is a Pasch geometry $\{\mathcal{S}, \mathcal{L}, d\}$ together with an angle measure $m$.

**<u>Definition</u> (Euclidean angle measure)** In the Euclidean Plane, the Euclidean angle measure of $\measuredangle ABC$ is

$$m_E(\measuredangle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\|\cdot\|C-B\|}\right)$$

**1.** In $\mathcal{E}$ what is $m_E(\measuredangle ABC)$ if $A(0,3)$, $B(0,1)$ and $C = (\sqrt{3}, 2)$?

**<u>Proposition</u>** $m_E$ is an angle measure on $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

**<u>Definition</u> (Euclidean tangent, Euclidean tangent ray)** If $\overrightarrow{BA}$ is a ray in the Poincaré Plane where $B(x_B, y_B)$ and $A(x_A, y_A)$, then the Euclidean tangent to $\overrightarrow{BA}$ at B is $T_{AB} =$

$$\begin{cases} (0, y_A - y_B), & \text{if } \overleftrightarrow{AB} \text{ is a type I line} \\ (y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B < x_A \\ -(y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B > x_A. \end{cases}$$

The Euclidean tangent ray to $\overrightarrow{BA}$ is the Euclidean ray $\overrightarrow{BA'}$ where $A' = B + T_{BA}$.

**<u>Definition</u> (Poincaré angle measure)** The measure of the Poincaré angle $\measuredangle ABC$ in $\mathbb{H}$ is

$$m_H(\measuredangle ABC) = m_E(\measuredangle A'BC') = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\|\cdot\|T_{BC}\|}\right)$$

**2.** In the Poincaré Plane find the measure of $\measuredangle ABC$ where $A(0,1)$, $B(0,5)$, and $C(3,4)$.

**<u>Proposition</u>** $m_H$ is an angle measure on $(\mathbb{H}, \mathcal{L}_H, d_H)$.

**3.** Let $A(0,1)$, $B(0,5)$, and $C(3,4)$ be points in the Poincaré Plane $\mathcal{H}$. Find the sum of the measures of the angles of $\triangle ABC$.

**<u>Convention</u>**. From now on the terms Euclidean Plane, Poincaré Plane, and Taxicab Plane will refer to the protractor geometries $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E, m_E\}$, $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H, m_H\}$, $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T, m_E\}$.